How Plot Digitizer's Calibration Works

Plot Digitizer allows the user to calibrate an image using any three non-collinear points. The calibration is a transformation of the image data between two reference frames. The reference frame of the scanned image is defined as the S' frame, and the reference frame of the computer monitor is defined as the S frame (see Figure 1 below). To make the transformation completely general, the scanned image may be rotated $(\theta \neq 0)$ and the axes of the image may be non-orthogonal $(\phi \neq 90^{\circ})$.

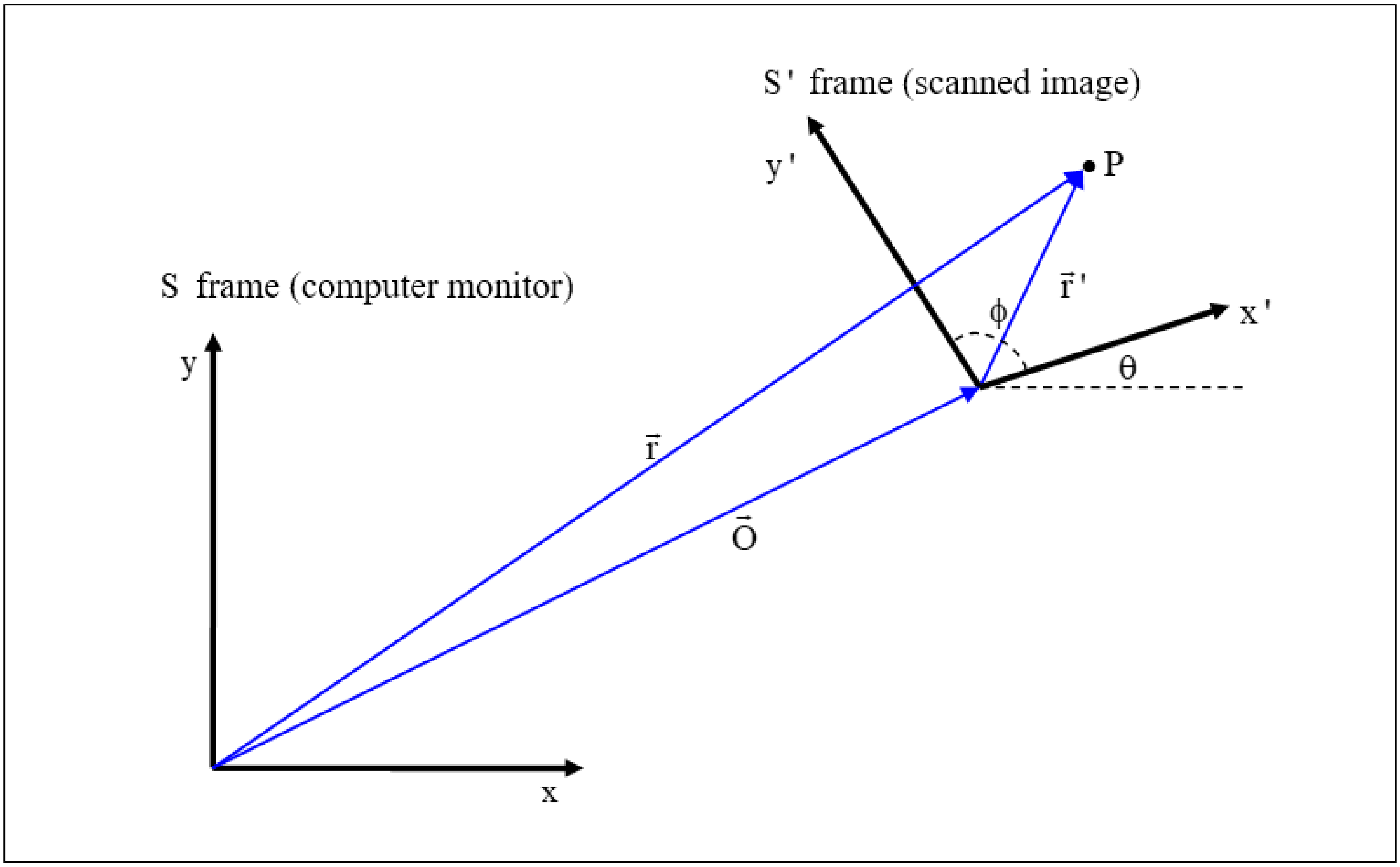

Figure 1: The reference frames.

The position of some point P in the S' frame is given by the position vector $\vec{r}\,' = x'\hat{i}' + y'\hat{j}'$. Writing the unit vectors $\hat{i}'$ and $\hat{j}'$ in terms of the S frame unit vectors $\hat{i}$ and $\hat{j}$ gives:

$$\hat{i}' = \cos(\theta)\,\hat{i} + \sin(\theta)\,\hat{j}$$
$$\hat{j}' = \cos(\phi+\theta)\,\hat{i} + \sin(\phi+\theta)\,\hat{j}$$

Therefore, $\vec{r}\,'$ written in terms of the S frame unit vectors is:

$$\vec{r}\,' = \left[x'\cos(\theta) + y'\cos(\phi+\theta)\right]\hat{i} + \left[x'\sin(\theta) + y'\sin(\phi+\theta)\right]\hat{j}$$

Notice from Figure 1 that the position of the point P in the S frame is given by $\vec{r} = \vec{O} + k\vec{r}\,'$ where $\vec{O}$ is a vector pointing to the origin of the S' frame, and k is a scale factor to transform the physical units of the scanned image to the dimensionless pixels of the computer monitor.

Finally, we can write the equations for transforming the (x', y') image coordinates of the point P to the (x, y) coordinates of the computer monitor.

$$x = O_x + kx'\cos(\theta) + ky'\cos(\phi+\theta)$$
$$y = O_y + kx'\sin(\theta) + ky'\sin(\phi+\theta)$$

Note that we have 2 equations with 6 unknowns $\left(O_x, O_y, \cos(\theta), \cos(\phi+\theta), \sin(\theta), \sin(\phi+\theta)\right)$. To solve for the 6 unknowns, we must therefore calibrate the image by selecting 3 points. This will give us 6 equations and 6 unknowns.

Writing out all 6 equations we get:

$$x_1 = O_x + kx_1'\cos(\theta) + ky_1'\cos(\phi+\theta)$$
$$y_1 = O_y + kx_1'\sin(\theta) + ky_1'\sin(\phi+\theta)$$

$$x_2 = O_x + kx_2'\cos(\theta) + ky_2'\cos(\phi+\theta)$$
$$y_2 = O_y + kx_2'\sin(\theta) + ky_2'\sin(\phi+\theta)$$

$$x_3 = O_x + kx_3'\cos(\theta) + ky_3'\cos(\phi+\theta)$$
$$y_3 = O_y + kx_3'\sin(\theta) + ky_3'\sin(\phi+\theta)$$

The points (x_1', y_1'), (x_2', y_2'), and (x_3', y_3') are the physical values (entered by the user during the calibration procedure) of the three calibration points. The points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are the pixel coordinates of the mouse as recorded by the computer when the user clicks the point during the calibration.

To make solving this set of equations a bit less cumbersome, I performed a change of variable:

$$a = k\cos(\theta)$$
$$b = k\cos(\phi+\theta)$$
$$c = O_x$$
$$d = k\sin(\theta)$$
$$e = k\sin(\phi+\theta)$$
$$f = O_y$$

This gives:

$$x_1 = x_1'a + y_1'b + c$$
$$y_1 = x_1'd + y_1'e + f$$

$$x_2 = x_2'a + y_2'b + c$$
$$y_2 = x_2'd + y_2'e + f$$

$$x_3 = x_3'a + y_3'b + c$$
$$y_3 = x_3'd + y_3'e + f$$

Writing the set of equations as a matrix equation $(\mathbf{Ax}=\mathbf{B})$ in terms of the redefined variables gives:

$$\begin{bmatrix} x_1' & y_1' & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1' & y_1' & 1 \\ x_2' & y_2' & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2' & y_2' & 1 \\ x_3' & y_3' & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3' & y_3' & 1 \end{bmatrix} \bullet \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$

Using Maple® to solve the set of equations, I finally get:

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \frac{1}{D} \begin{bmatrix} y_1'x_3 - y_1'x_2 + y_3'x_2 - y_2'x_3 + y_2'x_1 - y_3'x_1 \\ -x_1'x_3 - x_3'x_2 + x_2'x_3 + x_3'x_1 + x_1'x_2 - x_2'x_1 \\ y_1'x_3'x_2 - x_1y_2'x_3' - y_3'x_1'x_2 + y_3'x_2x_1 + x_3y_2'x_1' - y_1'x_2'x_3 \\ y_3'y_2 + y_2'y_1 + y_1'y_3 - y_3'y_1 - y_2'y_3 - y_1'y_2 \\ -x_2'y_1 + x_2'y_3 + x_3'y_1 - y_3x_1' - y_2x_3' + y_2x_1' \\ -x_2'y_1'y_3 + x_2'y_3'y_1 - x_3'y_2'y_1 + y_3y_2'x_1' - y_2y_3'x_1' + x_3'y_1'y_2 \end{bmatrix}$$

where $D = -y_2'x_3' + y_2'x_1' - y_3'x_1' - y_1'x_2' + y_2'x_2' - y_1'x_3'$.

The variables $a-f$ are calculated at the end of the calibration procedure when the user clicks the “Calibrate” button in the calibration pop-up window.

Finally, to get the physical values (x', y') of a clicked point (x, y), I solve the pair of equations

$$x = x'a + y'b + c$$
$$y = x'd + y'e + f$$

for (x', y') to get:

$$y' = \frac{a(y-f) - d(x-c)}{ea - db}$$
$$x' = \frac{x - by' - c}{a}$$

These equations calculate (x', y') as the mouse is moved in the calibrated image and the results are displayed in the status bar at the bottom of the screen. When the user clicks a point to digitize, the value (x', y') is copied to the data table.

Although it is not necessary to know the values of ϕ and θ for Plot Digitizer to work, this information is given to the user as a tool to assess the quality of the scanned image and its calibration.

From the definitions of $a-f$, $\theta=\tan^{-1}\left(d/a\right)$ and $\phi=\tan^{-1}\left(e/b\right)-\theta$. To avoid possible division by zero errors, I used the trig. identities $\tan^{-1}(x)=\sin^{-1}\left(x/\sqrt{1+x^2}\right)$ and $\tan^{-1}(x)=\cos^{-1}\left(1/\sqrt{1+x^2}\right)$ to finally get:

$\boxed{\theta=\sin^{-1}\left(d/\sqrt{a^2+d^2}\right)}$ and $\boxed{\phi=\cos^{-1}\left(b/\sqrt{e^2+b^2}\right)-\theta}$.